Integral Calculus

Reduction formulae

1) Maity & ghosh
2) K. C. Pal
(Master guide to Maity-ghosh)

1) Find a reduction formulae for ~~integration~~ $\int \sin^n x\, dx$, where n is Positive integer, >1.

$\Rightarrow$ Let, $I_n = \int \sin^n x\, dx$

$\therefore \quad = \int \sin^{n-1} x \cdot \sin x\, dx$

$= -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x \cos^2 x\, dx$

$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x)\, dx$

$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx$

$= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$

or, $n I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$

or, $\boxed{I_n = -\dfrac{\sin^{n-1} x \cos x}{n} + \dfrac{n-1}{n} I_{n-2}}$

2) Find a reduction formulae for $\int_0^{\pi/2} \sin^n x\, dx$, where n is Positive integer, >1

$\Rightarrow$ Let, $I_n = \int_0^{\pi/2} \sin^n x\, dx$

$= \int_0^{\pi/2} \sin^{n-1} x \cdot \sin^n x\, dx$

$= ~~(n-1)~~ \left[\sin^{n-1} x (-\cos x)\right]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x\, dx$

$= (n-1) \int_0^{\pi/2} \sin^{n-2} x (1 - \sin^2 x)\, dx$

$= (n-1) \int_0^{\pi/2} \sin^{n-2} x\, dx - (n-1) \int_0^{\pi/2} \sin^n x\, dx$

or, $n I_n = (n-1) I_{n-2}$

or, $\boxed{I_n = \dfrac{(n-1) I_{n-2}}{n}}$

3) evaluate $\int_0^{\pi/2} \sin^{12} x \, dx$.

If $I_n = \int_0^{\pi/2} \sin^n x \, dx$

Then we have,

$$I_n = \frac{n-1}{n} I_{n-2} \quad \text{---(i)}$$

Here, $I_{12} = \int_0^{\pi/2} \sin^{12} x \, dx$

using (i) repeatedly we have,

$$I_{12} = \frac{11}{12} I_{10}$$

$$= \frac{11}{12} \cdot \frac{9}{10} I_8 = \frac{11}{12} \cdot \frac{9}{10} \cdot \frac{7}{8} I_6$$

$$= \frac{11}{12} \cdot \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} I_4 = \frac{11}{12} \cdot \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} I_2$$

$$= \frac{11}{12} \cdot \frac{9}{10} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} I_0$$

$$\therefore I_0 = \int_0^{\pi/2} \sin^0 x \, dx = \frac{\pi}{2}$$

$$\therefore I_{12} = \frac{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

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1) Find a reduction formula $\int_0^{\pi/2} \cos^n x \, dx$.

⇒ We have, $I_n = \int_0^{\pi/2} \sin^n x \, dx = \frac{(n-1)}{n} I_{n-2}$

$$\therefore I_n = \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \sin^n\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \cos^n x \, dx$$

$$= \frac{n-1}{n} I_{n-2}$$

2) Show that $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx = \frac{(n-1)(n-3) \ldots 3 \cdot 1}{n(n-2) \ldots 4 \cdot 2} \cdot \frac{\pi}{2}$, when n is even.

⇒ Let, $I_n = \int_0^{\pi/2} \sin^n x \, dx$

∴ we have,

$$I_n = \frac{n-1}{n} I_{n-2} \quad \text{—(i)}$$

from (i),

$$I_n = \frac{n-1}{n} I_{n-2}$$
$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$
$$- - - - -$$
$$I_4 = \frac{3}{4} I_2$$
$$I_2 = \frac{1}{2} I_0$$

$$I_0 = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

multiplying columnwise,

$$I_n = \frac{(n-1)(n-3)\ldots 3.1}{n(n-2)\ldots 4.2} \cdot I_0$$
$$= \frac{(n-1)(n-3)\ldots 3.1}{n(n-2)\ldots 4.2} \cdot \frac{\pi}{2}$$

also, $I_n = \int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \sin^n\left(\frac{\pi}{2} - x\right) dx = \int_0^{\pi/2} \cos^n x\, dx$

3) Show that $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx = \frac{(n-1)(n-3)\ldots 4.2}{n(n-2)(n-4)\ldots 3.1} \cdot 1$, when n is odd.

⇒ Let, $I_n = \int_0^{\pi/2} \sin^n x\, dx$

we have, $I_n = \frac{n-1}{n} I_{n-2}$ —(i)

from (i)

$$I_n = \frac{n-1}{n} I_{n-2}$$
$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$
$$- - - - -$$
$$I_5 = \frac{4}{5} I_3$$
$$I_3 = \frac{2}{3} \cdot I_1$$

$$I_1 = \int_0^{\pi/2} \sin x\, dx = [-\cos x]_0^{\pi/2} = -[0-1] = 1$$

multiplying columnwise,

$$I_n = \frac{(n-1)(n-3)\ldots 4.2}{n(n-2)\ldots 5.3} I_1 = \frac{(n-1)(n-3)\ldots 4.2}{n(n-2)\ldots 5.3} \cdot 1$$

4) Evaluate $\int_0^{\pi/2} \cos^{16} x \, dx$.

$\Rightarrow \int_0^{\pi/2} \cos^{16} x \, dx = \frac{15.13.11.9.7.5.3.1}{16.14.12.10.8.6.4.2} \cdot \frac{\pi}{2}$

5) evaluate $\int_0^{\pi/2} \sin^{11} x \, dx$

$\Rightarrow \int_0^{\pi/2} \sin^{11} x \, dx = \frac{10.8.6.4.2.}{11.9.7.5.3.1} \cdot 1$

6) Find a ~~dot~~ reduction formula for $\int \sin^m x \cdot \cos^n x \, dx$; $m, n > 1$ are integers.

$\Rightarrow$ Let, $I_{m,n} = \int \sin^m x \cdot \cos^n x \, dx$

$$= \int \cos^{n-1} x \cdot (\sin^m x \cdot \cos x) \, dx$$

$$= \cos^{n-1} x \cdot \frac{\sin^{m+1} x}{m+1} - \int (n-1) \cos^{n-2} x (-\sin x) \frac{\sin^{m+1} x}{m+1} dx$$

$$= \frac{\cos^{n-1} x \cdot \sin^{m+1} x}{m+1} + \frac{(n-1)}{m+1} \int \cos^{n-2} x \sin^m x (1 - \cos^2 x) \, dx$$

$$= \frac{\cos^{n-1} x \cdot \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x \, dx - \frac{n-1}{m+1} \int \cos^n x \sin^m x \, dx$$

$$= \frac{\cos^{n-1} x \cdot \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{n-1}{m+1} I_{m,n}$$

or, $\left(1 + \frac{n-1}{m+1}\right) I_{m,n} = \frac{m+n}{m+1} I_{m,n} = \frac{\cos^{n-1} x \cdot \sin^{m+1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$

or, $I_{m,n} = \frac{\cos^{n-1} x \cdot \sin^{m+1} x}{m+n} + \frac{n-1}{m+n} I_{m,n-2}$. ——(i)

Similarly,

$I_{m,n} = - \frac{\cos^{n+1} x \cdot \sin^{m-1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n}$ ——(ii)

Note:- if $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$, then $I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$ —(iii)

$= \frac{m-1}{m+n} I_{m-2,n}$ —(iv)

7) evaluate $\int_0^{\pi/2} \sin^5 x \cos^7 x \, dx$.

⇒ Let, $I_{5,7} = \int_0^{\pi/2} \sin^5 x \cos^7 x \, dx$

we have the reduction formula,

$\frac{n-1}{m+n} I_{m,n-2}$ —(i)

and $\frac{m-1}{m+n} I_{m-2,n}$ —(ii)

∴ using (i),

$I_{5,7} = \frac{6}{12} I_{5,5}$

$I_{5,5} = \frac{4}{10} I_{5,3}$

$I_{5,3} = \frac{2}{8} I_{5,1}$

$I_{5,1} = \int_0^{\pi/2} \sin^5 x \cos x \, dx$

$= \left[\frac{\sin^6 x}{6}\right]_0^{\pi/2}$

$= \frac{1}{6}$

$I_{5,7} = \frac{6.4.2}{12.10.8} I_{5,1}$

$= \frac{6.4.2}{12.10.8} \cdot \frac{1}{6}$

8) evaluate $\int_0^{\pi/2} \sin^{10} x \cos^8 x \, dx$

⇒ $I_{10,8} = \int_0^{\pi/2} \sin^{10} x \cos^8 x \, dx$

we have,

$I_{10,8} = \frac{7}{18} I_{10,6}$

$= \frac{7}{18} \cdot \frac{5}{16} I_{10,4}$

$= \frac{7}{18} \cdot \frac{5}{16} \cdot \frac{3}{14} I_{10,2}$

$= \frac{7}{18} \cdot \frac{5}{16} \cdot \frac{3}{14} \cdot \frac{1}{12} \cdot I_{10,0}$

∴ $I_{10,0} = \int_0^{\pi/2} \sin^{10} x \, dx$

$= \frac{9.7.5.3.1}{10.8.6.4.2} \cdot \frac{\pi}{2}$

∴ $I_{10,8} = \frac{7.5.3.1.9.7.5.3}{18.16.14.12.10.8.6.4.2} \cdot \frac{\pi}{2}$

9) Prove the wallis' formula $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

$$= \frac{(m-1)(m-3)\ldots(1 \text{ or } 2)(n-1)(n-3)\ldots(1 \text{ or } 2)}{(m+n)(m+n-2)\ldots(2 \text{ or } 1)} \alpha$$

m, n are both integers, >1

$\alpha = \frac{\pi}{2}$, m and n both are even
$= 1$, for all other cases.

Q) evaluate $\int_0^{\pi/2} \sin^{12} x \cos^{10} x \, dx$.

⇒ using wallis' formula we have,

$$\int_0^{\pi/2} \sin^{12} x \cos^{10} x \, dx = \frac{11.9.7.5.3.1.9.7.5.3.1}{22.20.18.16.14.12.10.8.6.4.2} \times \frac{\pi}{2}$$

11) If $I_{m,n} = \int_0^1 x^m (1-x)^n \, dx$, then show that $(m+n+1) I_{m,n} = n I_{m,n-1}$ and hence deduce that $I_{m,n} = \frac{\lfloor m \lfloor n}{\lfloor m+n+1}$

⇒ We have,

$$I_{m,n} = \int_0^1 x^m (1-x)^n \, dx$$

$$= \left[\frac{x^{m+1}}{m+1}(1-x)^n\right]_0^1 - \int_0^1 \frac{x^{m+1}}{m+1} n (1-x)^{n-1} (-1) \, dx$$

$$= \frac{n}{m+1} \int_0^1 x^m (1-x)^{n-1} (1 - \overline{1-x}) \, dx$$

$$= \frac{n}{m+1} \int_0^1 x^m (1-x)^{n-1} dx - \frac{n}{m+1} \int x^m (1-x)^n dx$$

$$= \frac{n}{m+1} I_{m,n-1} - \frac{n}{m+1} I_{m,n}$$

or, $\left(\frac{m+n+1}{m+1}\right) I_{m,n} = \frac{n}{m+1} I_{m,n-1}$

or, $(m+n+1) I_{m,n} = n I_{m,n-1}$ ——— (i)

from (i) we have,

$$I_{m,n} = \frac{n}{m+n+1} I_{m,n-1} \quad \text{——— (ii)}$$

12)
13)
14)
15)
16)
17)
18)

$\therefore I_{m,n} = \frac{n}{m+n+1} I_{m,n-1}$

$I_{m,n-1} = \frac{n-1}{m+n} I_{m,n-2}$

$I_{m,n-2} = \frac{n-2}{m+n-1} I_{m,n-3}$

— — — — — — —

$I_{m,3} = \frac{3}{m+4} I_{m,2}$

$I_{m,2} = \frac{2}{m+3} I_{m,1}$

$I_{m,1} = \frac{1}{m+2} I_{m,0}$

multiplying columnwise,

$I_{m,n} = \frac{n(n-1)(n-2)\ldots 3.2.1}{(m+n+1)(m+n)\ldots(m+3)(m+2)} I_{m,0}$

$= \frac{\lfloor n}{(m+n+1)(m+n)\ldots(n+2)} I_{m,0}$

Now, $I_{m,0} = \int_0^1 x^m dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}$

$\therefore \frac{\lfloor n . \lfloor m}{(m+n+1)(m+n)(n+2)\ (m+1). \lfloor m} = \frac{\lfloor n . \lfloor m}{\lfloor m+n+1}$

12) Find a reduction formula for $I_{m,n} = \int_0^{\pi/2} \cos^m x \cdot \sin nx\, dx$ and hence deduce that $I_{m,m} = \frac{1}{2^{m+1}}\left[2+\frac{2^2}{2}+\frac{2^3}{3}+\ldots\frac{2^m}{m}\right]$

13) Find a reduction formula for $\int \frac{dx}{(a+b\cos x)^n}$.

14) Find a reduction formula for $\int_0^{\pi/2} x^n \sin x\, dx$

15) Find a reduction formula for $\int_0^1 x^n \tan^{-1}x\, dx,\ n>2$

16) Find a reduction formula for $\int \sin^m x \cos nx\, dx$

17) Find a reduction formula for $\int \sin^m x \sin nx\, dx$

18) Prove that if n be a positive integer, then $\int_0^{\pi/2} \cos^n x \cos nx\, dx = \frac{\pi}{2^{n+1}}$

19) If $I_n = \int_0^{\pi/2} x^n \sin x \, dx$ $(n > 1)$ then, show that,
$I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

20) Find a reduction formula for $\int \tan^n x \, dx$, $\int \sec^n x \, dx$.

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20) $I_n = \int \tan^n x \, dx = \int \tan^{n-2} x \cdot \tan^2 x \, dx = ~~\tan^{n-2} x \sec x - \int (n-2) \tan x \sec x~~$

$= \int \tan^{n-2} x (\sec^2 x - 1) dx = \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$

$= \tan^{n-2} x \tan x - \int (n-2) \tan^{n-3} x \sec^2 x \tan x \, dx - I_{n-2}$

$= \tan^{n-1} x - (n-2) \int \tan^{n-2} x (1 + \tan^2 x) dx - I_{n-2}$

$= \tan^{n-1} x - (n-2) \int \tan^{n-2} x \, dx - (n-2) \int \tan^n x \, dx - I_{n-2}$

$= \tan^{n-1} x - (n-2) I_{n-2} - (n-2) I_n - I_{n-2}$

$(n-1) I_n = \tan^{n-1} x - (n-1) I_{n-2}$

or, $I_n = \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$

Let, $I_n = \int \sec^n x \, dx = \int \sec^{n-2} x \cdot \sec^2 x \, dx$

$= \sec^{n-2} x \tan x - \int (n-2) \sec^{n-3} x \sec x \tan^2 x \, dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) dx$

$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$

$= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2}$

$(n-1) I_n = \sec^{n-2} x \tan x + (n-2) I_{n-2}$

or, $I_n = \dfrac{\sec^{n-2} x \tan x + (n-2) I_{n-2}}{(n-1)}$.

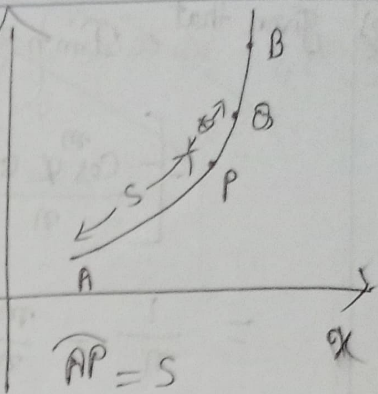
Arc Length

length of the arc AB $= \int_A^B ds$, where

i) $ds = \sqrt{1+\left(\frac{ds}{dx}\right)^2}\, dx$, for cartesian curve $y = f(x)$

ii) $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$, for polar curve $r = f(\theta)$

iii) $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$ for parametric curve $x = x(t)$, $y = y(t)$



$\widehat{AP} = S$

$\widehat{PQ} = ds$

1q) given that, $I_n = \int_0^{\pi/2} x^n \sin x\, dx \quad (n>1)$

$= \left[-x^n \cos x\right]_0^{\pi/2} + \int_0^{\pi/2} n x^{n-1} \cos x\, dx$

$= n \int_0^{\pi/2} x^{n-1} \cos x\, dx = \left[n x^{n-1} \sin x\right]_0^{\pi/2} - n\int_0^{\pi/2} (n-1) x^{n-2} \sin x\, dx$

$= +n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}$

or, $I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$ (Proved)

14) Let, $I_n = \int_0^{\pi/2} x^n \sin x\, dx = \left[-x^n \cos x\right]_0^{\pi/2} + n\int_0^{\pi/2} x^{n-1} \cos x\, dx$

$= \left[n x^{n-1} \sin x\right]_0^{\pi/2} - n(n-1)\int_0^{\pi/2} x^{n-2} \sin x\, dx$

$= \left[n x^{n-1} \sin x\right]_0^{\pi/2} - n(n-1) I_{n-2} \quad (i) = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}$

~~Now, from (i) we have,~~

~~$I_n = n x^{n-1} \sin x - n(n-1) I_{n-2}$~~

~~$I_{n-2} = (n-2) x^{n-3} \sin x - (n-2)(n-3) I_{n-4}$~~

12) given that, $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx \, dx$ —(i)

$$= \left[-\frac{\cos^m x \cos nx}{n}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{m \cos^{m-1} x \sin x}{n}(-\cos nx)\,dx$$

$$= \frac{1}{n} - \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x \{\cos x \sin nx - \sin(n-1)x\}\,dx$$

$$= \frac{1}{n} - \frac{m}{n}\int_0^{\pi/2} \cos^m x \sin nx\,dx + \frac{m}{n}\int_0^{\pi/2} \sin(n-1)x \cos^{m-1} x\,dx$$

$$= \frac{1}{n} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

or, $\frac{m+n}{n} I_{m,n} = \frac{1}{n} + \frac{m}{n} I_{m-1,n-1}$

or, $I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$ —(ii)

Now, from (ii) we have,

$$I_{m,m} = \frac{1}{2m} + \frac{m}{2m} I_{m-1,m-1}$$

$$= \frac{1}{2m} + \frac{1}{2}\left(\frac{1}{2m-2} + \frac{m-1}{2m-2} I_{m-2,m-2}\right)$$

~~$= \frac{1}{2^{m+1}}$~~ $\left[= \frac{1}{2m} + \frac{1}{2^2}\left(\frac{1}{m-1} + \frac{1}{2^2} I_{m-2,m-2}\right)\right]$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^2}\left(\frac{1}{2m-4} + \frac{m-2}{2m-4} I_{m-3,m-3}\right)$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \frac{1}{2^3} I_{m-3,m-3}$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \dots + \frac{1}{2^{m-1}} I_{1,1}$$ —(iii)

~~$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \dots + \frac{1}{2^{m-1}}\left(\frac{1}{2} + \frac{1}{2^2}\right)$~~

~~$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-3)} + \dots + \frac{1}{2^{m-1}}$~~

$\therefore$ from (i), $I_{1,1} = \int_0^{\pi/2} \cos x \sin x \, dx$ ~~$= \frac{1}{2}\int_0^{\pi/2} \sin 2x \, dx = -\frac{1}{4}[\cos 2x]_0^{\pi/2} = \frac{1}{4}$~~

$\therefore$ from (iii), $= [-\sin x]_0^{\pi/2} = -1$

$$I_{m,m} = \frac{1}{2m} + \frac{1}{2^2(m-1)} + \dots + \frac{1}{2^{m-1}} \cdot \frac{1}{2^2}$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \dots + \frac{1}{2^{m+1}}$$

16) Let, $I_{m,n} = \int \sin^m x \cos nx \, dx$

$$= \frac{\sin^m x \sin nx}{n} - \frac{m}{n}\int \sin^{m-1} x \sin nx \cos x \, dx$$

$$= \frac{\sin^m x \sin nx}{n} - \frac{m}{n}\int \sin^{m-1} x \{\sin x \cos nx + \sin(n-1)x\} dx$$

$$= \frac{\sin^m x \sin nx}{n} - \frac{m}{n}\int \sin^m x \cos nx \, dx - \frac{m}{n}\int \sin^{m-1} x \sin(n-1)x \, dx$$

$$= \frac{\sin^m x \sin nx}{n} - \frac{m}{n} I_{m,n} - \frac{m}{n} I_{m-1,n-1}$$

or, $\frac{m+n}{n} I_{m,n} = \frac{\sin^m x \sin nx}{n} - \frac{m}{n} I_{m-1,n-1}$

or, $I_m = \frac{\sin^m x \sin nx}{m+n} - \frac{m}{m+n} I_{m-1,n-1}$

17) Let, $I_{m,n} = \int \sin^m x \sin nx \, dx = -\frac{\sin^m x \cos nx}{n} + \frac{m}{n}\int \sin^{m-1} x \cos x \cos nx \, dx$

$$= -\frac{\sin^m x \cos nx}{n} + \frac{m}{n}\int \sin^{m-1} x \{\cos(n-1)x - \sin x \sin nx\} dx$$

$$= -\frac{\sin^m x \cos nx}{n} + \frac{m}{n}\int \sin^{m-1} x \cos(n-1)x \, dx - \frac{m}{n}\int \sin^m x \sin nx \, dx$$

$$= -\frac{\sin^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1} - \frac{m}{n} I_{m,n}$$

or, $\frac{m+n}{n} I_{m,n} = -\frac{\sin^m x \cos nx}{n} + \frac{m}{n} I_{m-1,n-1}$

or, $I_{m,n} = -\frac{\sin^m x \cos nx}{m+n} + \frac{m}{m+n} I_{m-1,n-1}$.

Q. 18) Let, $I_{n,n} = \int_0^{\pi/2} \cos^n x \cos nx \, dx$ ——(i)

$= \left[\frac{\cos^n x \sin nx}{n}\right]_0^{\pi/2} + \frac{n}{n}\int_0^{\pi/2} \cos^{n-1} x \sin x \sin nx \, dx$

$= \int_0^{\pi/2} \cos^{n-1} x \{\cos(n-1)x - \cos x \cos nx\} dx$

$= \int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x \, dx - \int_0^{\pi/2} \cos^n x \cos nx \, dx$

$2 I_{n,n} = I_{n-1,n-1}$

or, $I_{n,n} = \frac{I_{n-1,n-1}}{2} = \frac{I_{0,0}}{2^n}$ ——(ii) .... $= \frac{I_{0,0}}{2^n}$

from (i), we get,

$I_{0,0} = \int_0^{\pi/2} \cos x \, dx = [\sin x]_0^{\pi/2} = \frac{\pi}{2}$

∴ from (ii) we have,

$I_{n,n} = \frac{\pi}{2 \cdot 2^n} = \frac{\pi}{2^{n+1}}$ (Proved)

19) Let, $I_n = \int_0^1 x^n \tan^{-1} x \, dx$

$= \left[\frac{\tan^{-1} x \, x^{n+1}}{n+1}\right]_0^1 - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2} dx$

$= \frac{\pi}{4(n+1)} - \frac{1}{n+1}\int_0^1 \frac{x^2}{1+x^2} \cdot x^{n-1} dx$

or, $(n+1) I_n = \frac{\pi}{4} - \int_0^1 \frac{1+x^2-1}{1+x^2} x^{n-1} dx$

$= \frac{\pi}{4} - \int_0^1 x^{n-1} dx + \int_0^1 \frac{x^{n-1}}{1+x^2} dx$

$= \frac{\pi}{4} - \left[\frac{x^n}{n}\right]_0^1 + \left(x^{n-1} \tan^{-1} x\right)_0^1 - \int_0^1 (n-1) x^{n-2} \tan^{-1} x \, dx$

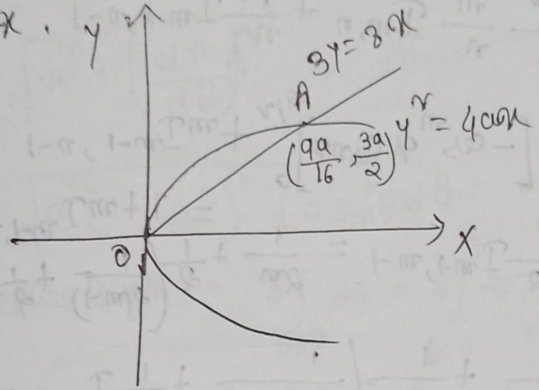
$= \frac{\pi}{4} - \frac{1}{n} + \frac{\pi}{4} - (n-1)\int_0^1 x^{n-2} \tan^{-1} x \, dx$

$= \frac{\pi}{2} - \frac{1}{n} - I_{n-2}(n-1)$

or, $(n+1)I_n + (n-1) I_{n-2} = \frac{\pi}{2} - \frac{1}{n}$ (Proved)

Arc length

1) Find the length of the arc of the parabola $y^2 = 4ax$ cut off by $3y = 8x$.



We have, $y^2 = 4ax$

or, $2y\frac{dy}{dx} = 4a$

or, $\frac{dy}{dx} = \frac{2a}{y}$

∴ $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\frac{4a^2}{y^2}}\,dx$

$= \sqrt{\frac{y^2+4a^2}{y^2}}\,dx = \sqrt{\frac{4ax+4a^2}{4ax}}\,dx = \sqrt{\frac{x+a}{x}}\,dx$

∴ The required length $OA = \int_0^A ds = \int_0^{\frac{9a}{16}} \sqrt{\frac{x+a}{x}}\,dx$

~~$= \int_0^{\frac{9a}{16}} dx + \sqrt{a}\int_0^{\frac{9a}{16}} x^{-1/2}\,dx$~~

~~$= \frac{9a}{16} + 2\sqrt{a}\left[x^{1/2}\right]_0^{\frac{9a}{16}} = \frac{9a}{16} + 2\sqrt{a}\left(\frac{9a}{16}\right)^{1/2}$~~

~~$= \frac{9a}{16} + 2\sqrt{a}\,\frac{3\sqrt{a}}{4} = \frac{9a}{16} + \frac{3}{2}a$~~

$= \frac{9a+24a}{16} = \frac{33a}{16}$

Let, $x = az^2$

x	0	$\frac{9a}{16}$
z	0	$\frac{3}{4}$

$dx = 2az\,dz$

∴ $\int_0^{\frac{3}{4}} \sqrt{\frac{a+az^2}{az^2}}\,2az\,dz$

$= \frac{15a}{16} + a\log 2$

9 12) Let, $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx \, dx$

$$= \left[-\frac{\cos^m x \cos nx}{n}\right]_0^{\pi/2} - \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x \cdot \sin x \cos nx \, dx$$

$$= \left(-\frac{\cos^m x \cos nx}{n}\right)_0^{\pi/2} - \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x \sin nx \cos x \, dx + \frac{m}{n}\int \cos^{m-1} x \sin (n-1)x \, dx$$

$$= \left[-\frac{\cos^m x \cos nx}{n}\right]_0^{\pi/2} - \frac{m}{n} I_{m,n} + \frac{m}{n} I_{m-1,n-1}$$

or, $(m+n) I_{m,n} = \left[-\cos^m x \cos nx\right]_0^{\pi/2} + m I_{m-1,n-1}$

$$= 1 + m I_{m-1,n-1}$$

$$I_{m,m} = \frac{1}{2m} + \frac{1}{2} I_{m-1,m-1} = \frac{1}{2m} + \frac{1}{2}\left(\frac{1}{2(m-1)} + \frac{1}{2} I_{m-2,m-2}\right)$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^2}\left(\frac{1}{2(m-2)} + \frac{1}{2} I_{m-3,m-3}\right)$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \frac{1}{2^3} I_{m-3,m-3}$$

after $(m-1)$ times,

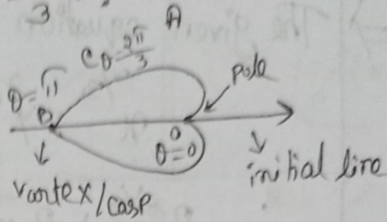
$$I_{m,m} = \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \dots + \frac{1}{2^{m-2} \cdot 3} + \frac{1}{2^{m-1} \cdot 2} + \frac{1}{2^{m-1}} I_{1,1}$$

$$= \frac{1}{2m} + \frac{1}{2^2(m-1)} + \frac{1}{2^3(m-2)} + \dots + \frac{1}{2^{m-2} \cdot 3} + \frac{1}{2^{m-1} \cdot 3} + \frac{1}{2^{m-1} \cdot 2} + \frac{1}{2^{m-1} \cdot 2}$$

Since, $I_{1,1} = \int_0^{\pi/2} \cos x \sin x \, dx = \frac{1}{2}$

$$\therefore I_{m,m} = \frac{1}{2^{m+1}}\left(2 + \frac{2^2}{2} + \frac{2^3}{2} + \dots + \frac{2^{m-1}}{m-1} + \frac{2^m}{m}\right)$$

2) Find the length of the cardiode $r = a(1-\cos\theta)$ and show that the arc of the upper half of the curve is bisected by $\theta = \frac{2\pi}{3}$.



⇒ the given equation is

$r = a(1-\cos\theta)$ —— (i)

$\therefore \frac{dr}{d\theta} = a\sin\theta$

$\therefore ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

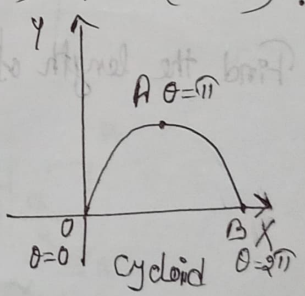
$= \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta = a\sqrt{2(1-\cos\theta)}\, d\theta = 2a\sin\frac{\theta}{2}\, d\theta$

$\therefore$ The require length $= 2\int_{OAB} ds = 2\int_{\theta=0}^{\pi} 2a\sin\theta/2\, d\theta = 4a\int_{\theta=0}^{\pi} \sin\theta/2\, d\theta$

$= -8a\left[\cos\frac{\theta}{2}\right]_0^{\pi} = -8a[0-1] = 8a$

$\therefore \widehat{OC} = \int ds = \int_{\theta=0}^{2\pi/3} 2a\sin\frac{\theta}{2}\, d\theta = 2a\left[-2\cos\frac{\theta}{2}\right]_0^{\frac{2\pi}{3}} = -4a\left[\cos\frac{\pi}{3} - 1\right] = -4a\left[\frac{1}{2}-1\right] = \frac{4a}{2} = 2a$

3) Find the length of one arch of $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$.



⇒ $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$

$\therefore \frac{dx}{d\theta} = a(1-\cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$

$\therefore ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

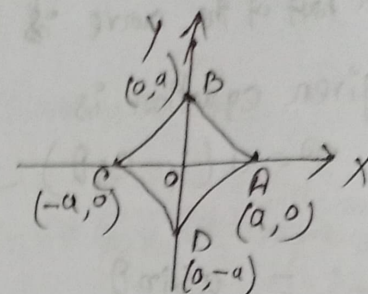
$= \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta = 2a\sin\theta/2\, d\theta$

$\therefore$ The required length $= \int_{OAB} ds = 2a\int_{\theta=0}^{2\pi} \sin\theta/2\, d\theta = -4a\left[\cos\frac{\theta}{2}\right]_0^{2\pi}$

$= -4a[\cos\pi - 1] = 8a$

4) Find the length of perimeter of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$

⇒ The given equation,



$$x^{2/3}+y^{2/3}=a^{2/3} \quad \text{—(i)}$$

$$\therefore \frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0$$

$$\text{or, } \frac{dy}{dx}=-\left(\frac{y}{x}\right)^{1/3}$$

[~~OA = OB =~~]

[$\widehat{AB}=\widehat{BC}=\widehat{CD}=\widehat{DA}$]

$$\therefore ds=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+\frac{y^{2/3}}{x^{2/3}}}\,dx$$

$$=\sqrt{\frac{a^{2/3}}{x^{2/3}}}\,dx=a^{1/3}\cdot x^{-1/3}\,dx$$

$$-\frac{1}{3}+1=\frac{-1+3}{3}=\frac{2}{3}$$

$$\therefore \text{The required length}=4\int a^{1/3}x^{-1/3}\,dx$$

$$=4a^{1/3}\int_{0}^{a}x^{-1/3}\,dx=\frac{3\times4}{2}a^{1/3}\left[x^{2/3}\right]_0^a$$

BA

$$=6a^{1/3}a^{2/3}=6a.$$

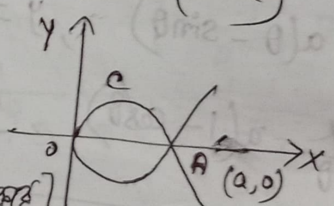
5) Find the length of the loop of the curve $3ay^2=x(x-a)^2$

$x=0,a,a$

↓

double Point

[∵ A Point কে 2বার cross করে]



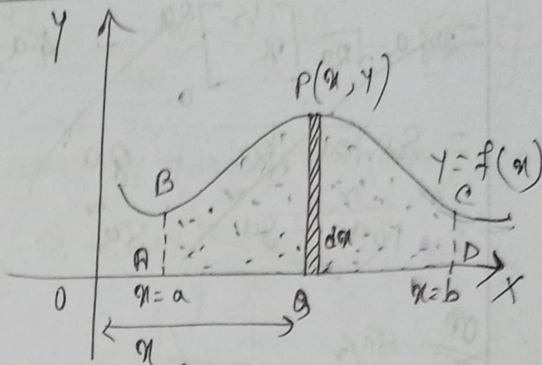
$$\left[\frac{dy}{dx}=\frac{(x-a)(3x-a)}{6ay}\right]$$

6) Find the length of the hypocycloid $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$

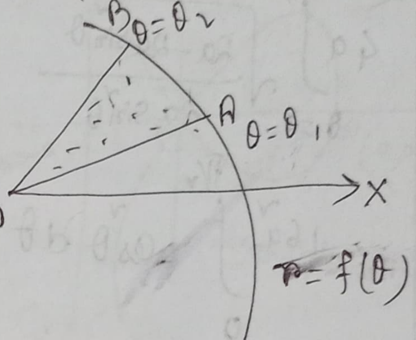
7) Find the length of the arc of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ intercepted between the coordinate axes.

Evaluation of Area

i) Area of ABCDA $= \int_{x=a}^{b} y\,dx$

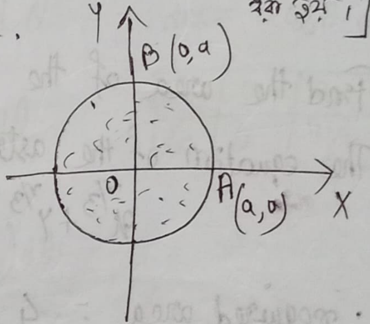


ii) Area of OABO $= \frac{1}{2}\int_{\theta=\theta_1}^{\theta_2} r^2\,d\theta$



[Polar coordinate system এ anticlock wise +ve sense ধরা হয়।]

1) Find the area of a circle of radius a.



⇒ Equation of the circle, $x^2+y^2=a^2$ —(i)

∴ required area $= 4 \times$ area of OABO

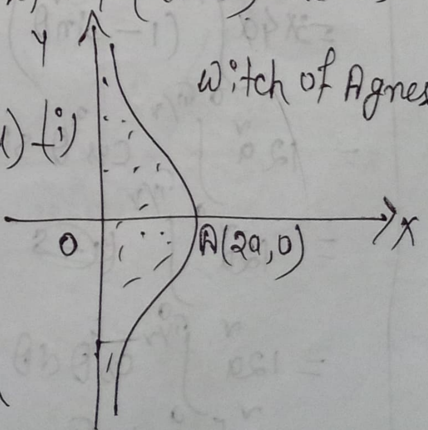
$= 4\int_0^a y\,dx$

$= 4\int_0^a \sqrt{a^2-x^2}\,dx$

$= 4\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$= 4\left[\frac{a^2}{2}\times\frac{\pi}{2}\right] = \pi a^2$

2) Find the area between the curve $xy^2 = 4a^2(2a-x)$ and it's asymptote.



⇒ Equation of the curve, $xy^2 = 4a^2(2a-x)$ —(i)

∴ The required area $= 2\int_0^{2a} y\,dx$

$= 2\int_{x=0}^{2a} 2a\sqrt{\frac{2a-x}{x}}\,dx$

$= 4a\int_0^{2a}\sqrt{2a}\cdot x^{-1/2}\,dx - 4a\int_0^{2a} dx$

$2\times 4a\sqrt{2a}\left[x^{1/2}\right]_0^{2a} - 4a[x]_0^{2a}$

$= 8a\sqrt{2a}\sqrt{2a} - 8a^2$

$= 16a^2 - 8a^2$

$8a^2$

or

$$4a\int_{\theta=0}^{\pi/2}\sqrt{\frac{2a-2a\sin^2\theta}{2a\sin^2\theta}}\,4a\sin\theta\cos\theta\,d\theta$$

$x = 2a\sin^2\theta$

$dx = 4a\sin\theta\cos\theta\,d\theta$

x	0	$2a$
θ	0	$\frac{\pi}{2}$

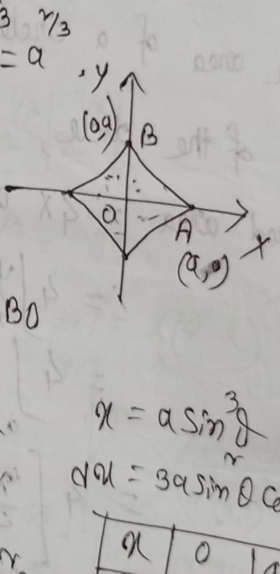
$$= 16a^2\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$= 16a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 4\pi a^2$$

3) Find the area of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$

⇒ The equation of the astroid,

$$x^{2/3}+y^{2/3}=a^{2/3} \quad \text{—(i)}$$



∴ required area $= 4\times$ area of OABO

$$= 4\int_{x=0}^{a}\left(a^{2/3}-x^{2/3}\right)^{3/2}dx$$

$x = a\sin^3\theta$

$dx = 3a\sin^2\theta\cos\theta\,d\theta$

x	0	a
θ	0	$\pi/2$

$$= 4\int_0^{\pi/2}\left(a^{2/3}-a^{2/3}\sin^2\theta\right)^{3/2}3a\sin^2\theta\cos\theta\,d\theta$$

$$= 3\times 4a^2\int_0^{\pi/2}(1-\sin^2\theta)^{3/2}\sin^2\theta\cos\theta\,d\theta$$

$$= 12a^2\int_0^{\pi/2}\cos^3\theta\sin^2\theta\cos\theta\,d\theta = 12a^2\int_0^{\pi/2}\cos^4\theta\sin^2\theta\,d\theta$$

$$= 12a^2\int_0^{\pi/2}\cos^4\theta(1-\cos^2\theta)\,d\theta$$

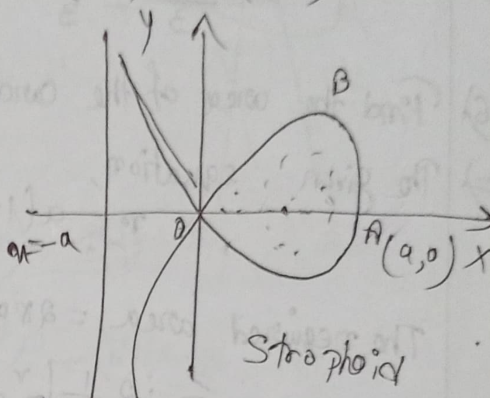
$$= 12a^2\int_0^{\pi/2}\cos^4\theta\,d\theta - 12a^2\int_0^{\pi/2}\cos^6\theta\,d\theta$$

$$= 12a^2\left[\frac{3.1}{4.2} - \frac{5.3}{6.4.2}\right]\frac{\pi}{2} = \frac{36a^2\cdot 3\pi}{4\cdot 2}\left[3-\frac{35}{6}\right]$$

$= \frac{9a^2\pi}{2 \cdot 6a^2\pi \times \frac{4 \cdot 3}{8}\left[1 - \frac{5}{6}\right]} \times \frac{18-35}{6} = \frac{3 \cdot 17 \times 17\, a^2\pi}{4 \times 6^2} = \frac{3\pi a^2}{8^2} = \frac{51\pi a^2}{8}$ $\left[\frac{3}{8}\pi a^2\right]$

4) Find the area of the loop of the curve $y^2(a+x) = x^2(a-x)$, $a>0$

⇒ required area, $2 \times$ area of OABO or, $x(x^2+y^2) = a(x^2-y^2)$



$$= 2\int_0^a y\,dx$$

$$= 2\int_{x=0}^{a} x\sqrt{\frac{a-x}{a+x}}\,dx$$

$$= 2\int_0^a x\,\frac{a-x}{\sqrt{a^2-x^2}}\,dx$$

$$= 2\int_0^{\pi/2} a\sin\theta\,\frac{a(1-\sin\theta)}{a\cos\theta}\,a\cos\theta\,d\theta \qquad \left[2a^2\left(1-\frac{\pi}{4}\right)\right]$$

$x = a\sin\theta$

$dx = a\cos\theta\,d\theta$

x	0	a
θ	0	$\pi/2$

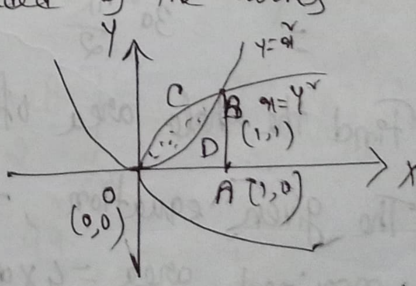
$$= 2a^2\int_0^{\pi/2}\sin\theta\,d\theta - 2a^2\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= \left[-2a^2\cos\theta\right]_0^{\pi/2} - a^2\int_0^{\pi/2}(1-\cos 2\theta)\,d\theta$$

$$= 2a^2 - a^2\left[\theta - \frac{\sin 2\theta}{\theta}\right]_0^{\pi/2} = 2a^2 - a^2\left[\frac{\pi}{2}\right]$$

$$= 2a^2\left(1 - \frac{\pi}{4}\right)$$

5) Find the area of the region bounded by the curves $y = x^2$ and $x = y^2$



⇒ The given equations, $y = x^2$ —(i)

$x = y^2$ —(ii)

from (i) and (ii),

$x = x^4$

or, $(x - x^4) = 0$

or, $x = 1, 0$

$\therefore y = 1, 0.$

∴ The required area = area of ODBCO = area of OABCO − area of OABCO

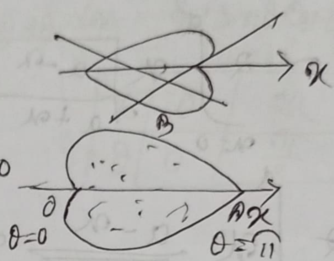
$= \int y\,dx\,(\text{for } 1) - \int y\,dx\,(\text{for } 2) = \int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx = \frac{2}{3}\left[x^{3/2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1$

$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

6) Find the area of the cardoid $r = a(1+\cos\theta)$.

⇒ The given equation,

$r = a(1+\cos\theta)$



The required area = 2 × area of OBAO

$= 2 \cdot \frac{1}{2}\int r^2\,d\theta$

$= \int_{\theta=0}^{\pi} a^2(1+\cos\theta)^2\,d\theta$

~~$= a^2\int_0^{\pi} 1+\cos^2\theta + 2\cos\theta\,d\theta$~~

~~$= a^2\int_0^{\pi} d\theta + \frac{a^2}{2}\int_0^{\pi}(1+\cos 2\theta)\,d\theta + 2a\int_0^{\pi/2}\cos\theta\,d\theta$~~

$= 4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta$

$\frac{\theta}{2} = \phi$

$d\theta = 2\,d\phi$

θ	0	π
φ	0	π/2

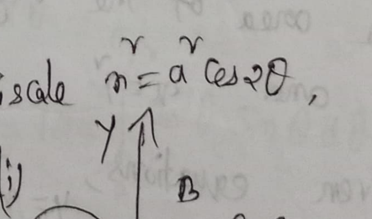
$= 4a^2 \cdot 2\int_{\phi=0}^{\pi/2}\cos^4\phi\,d\phi$

$= 8a^2 \cdot \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}$

$= \frac{3a^2\pi}{2}$

7) Find the whole area of the Lamniscale $r^2 = a^2\cos 2\theta$,

⇒ The given equation, $r^2 = a^2\cos 2\theta$ —(i)



∴ required area = 4 × area of OABO

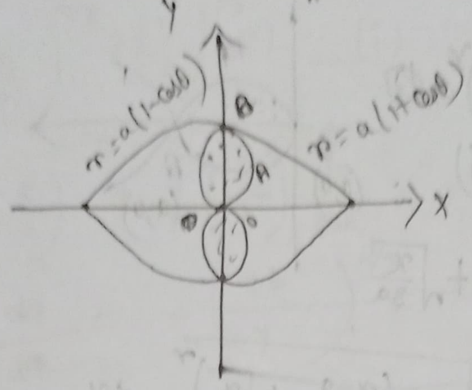
$= 4 \times \frac{1}{2}\int r^2\,d\theta$

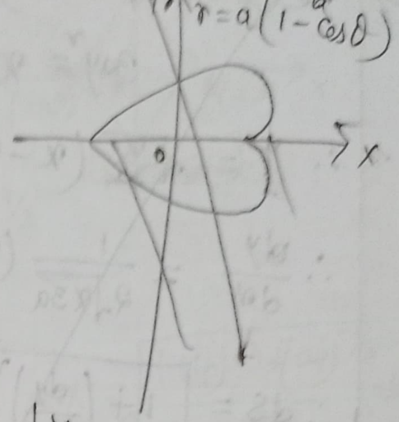
$= 2\int_{\theta=0}^{\pi/4} a^2\cos 2\theta\,d\theta = \frac{2a^2}{2}\left[\sin 2\theta\right]_0^{\pi/4} = a^2\left[\sin\frac{\pi}{2}\right]$

$= a^2$

8) Find the area included between in the cardoids $r = a(1+\cos\theta)$ and $r = a(1-\cos\theta)$

⇒





The equations are, $r = a(1+\cos\theta)$ ___ (i)

$r = a(1-\cos\theta)$ ___ (ii)

∴ The required area $= 4 \times$ area of OABO

$= 4 \times \frac{1}{2} \int_{\theta=0}^{\pi/2} r^2 \, d\theta$ (for the curve 2)

$= 2a^2 \int_0^{\pi/2} (1-\cos\theta)^2 \, d\theta = 2a^2 \int_0^{\pi/2} 4\sin^4 \theta/2 \, d\theta$

$= 8a^2 \int_0^{\pi/2} \sin^4 \theta/2 \, d\theta$

$= 2 \times 8a^2 \int_0^{\pi/4} \sin^4 \phi \, d\phi$

$= 16a^2 \int_0^{\pi/4}$

θ	0	$\pi/2$
ϕ	0	$\pi/4$

$\frac{\theta}{2} = \phi$

$d\theta = 2d\phi$

$= 2a^2 \int_0^{\pi/2} (1 + \cos^2\theta - 2\cos\theta) \, d\theta$

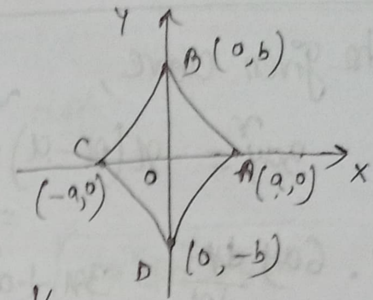
$= 2a^2 \int_0^{\pi/2} d\theta + a^2 \int_0^{\pi/2} (1+\cos 2\theta) \, d\theta - 4a^2 \int_0^{\pi/2} \cos\theta \, d\theta$

$= 2a^2 \frac{\pi}{2} + a^2 \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} - 4a^2 [\sin\theta]_0^{\pi/2}$

$= a^2\pi + a^2 \left[\frac{\pi}{2}\right] - 4a^2$

$= a^2\left(\pi + \frac{\pi}{2} - 4\right) = a^2\left(\frac{3\pi}{2} - 4\right)$

6) The given equation,



$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1 \quad \text{—(i)}$$

$$\therefore \frac{2}{3}\left(\frac{x}{a}\right)^{-1/3}+\frac{2}{3}\left(\frac{y}{b}\right)^{-1/3}\frac{dy}{dx}=0$$

$$\text{or, } \frac{dy}{dx}=-\left(\frac{xb}{ay}\right)^{-1/3}=-\left(\frac{ay}{xb}\right)^{1/3}$$

$$\therefore dS=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+\left(\frac{ay}{xb}\right)^{2/3}}\,dx=\sqrt{\frac{(xb)^{2/3}+(ay)^{2/3}}{(xb)^{2/3}}}\,dx$$

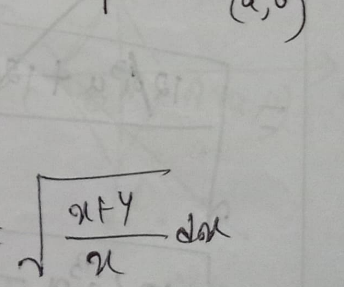
$$=\sqrt{\frac{\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}}{\left(\frac{x}{a}\right)^{2/3}}}\,dx=\sqrt{\frac{1}{\left(\frac{x}{a}\right)^{2/3}}}\,dx \quad [\text{by (i)}]$$

$$=\sqrt{\left(\frac{x}{a}\right)^{-2/3}}\,dx=\left(\frac{x}{a}\right)^{-\frac{1}{3}}dx$$

$\therefore$ The required length $= 2\int_0^a \left(\frac{x}{a}\right)^{-1/3}dx + 2 = 2a^{1/3}\int_0^a x^{-1/3}\,dx + 2a^{1/3}\int_0^b x^{-1/3}\,dx$

$$= -\frac{4}{3}\left[\left(\frac{x}{a}\right)^{-4/3}\right]_0^a = -\frac{4}{3}[0-1] = \frac{4}{3} = 3a + 3a^{1/3}b^{2/3}$$

7) The given equation,



$$\sqrt{x}+\sqrt{y}=\sqrt{a} \quad \text{—(i)}$$

$$\therefore \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\text{or, } \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

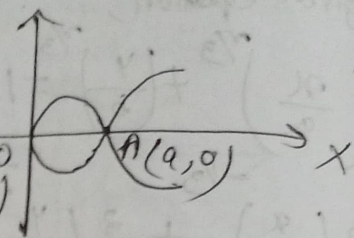
$$\therefore dS=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\sqrt{1+\frac{y}{x}}\,dx=\sqrt{\frac{x+y}{x}}\,dx$$

$$=\sqrt{\frac{x+(\sqrt{a}-\sqrt{x})^2}{x}}\,dx=\sqrt{\frac{x+a+x-2\sqrt{ax}}{x}}\,dx$$

$$=\sqrt{\frac{2x-2\sqrt{ax}+a}{x}}\,dx$$

1) The given curve,

$$3ay^2 = x(x-a)^2 = x(x^2+a^2-2ax)$$
$$= x^3 + a^2x - 2ax^2$$



$$\therefore 6ay\frac{dy}{dx} = 3x^2 + a^2 - 4ax$$

$$\text{or, } \frac{dy}{dx} = \frac{3x^2 - 4ax + a^2}{6ay}$$

$$= \frac{3x^2 - 3ax - ax + a^2}{6ay}$$

$$= \frac{3x(x-a) - a(x-a)}{6ay} = \frac{(x-a)(3x-a)}{6ay}$$

$$\therefore dS = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \sqrt{1 + \frac{(x-a)^2(3x-a)^2}{36a^2y^2}}\, dx$$

$$= \sqrt{\frac{12a(x^3 + a^2x - 2ax^2) + (3x^2 - 3xa - ax + a^2)^2}{12a(x^3 + a^2x - 2ax^2)}}\, dx$$

$$= \sqrt{\frac{12x^3a + 12a^3x - 24a^2x^2 + 9x^4 + a^4 + 16x^2a^2 - 12x^3a - 8xa^3 + 6x^2a^2}{12ax^3 + 12a^3x - 24a^2x^2}}\, dx$$

$$= \sqrt{\frac{4a^3x - 2a^2x^2 + 9x^4 + a^4}{12ax^3 + 12a^3x - 24a^2x^2}}\, dx$$

$$= \sqrt{\frac{36a^2y^2 + (x-a)^2(3x-a)^2}{36a^2y^2}}\, dx$$

$$= \sqrt{\frac{12ax(x-a)^2 + (x-a)^2(3x-a)^2}{12ax(x-a)^2}}\, dx$$

$$= \sqrt{\frac{12ax + 9x^2 + a^2 - 6ax}{12ax}}\, dx = \sqrt{\frac{9x^2 + a^2 + 6ax}{12ax}}\, dx$$

$$= \sqrt{\frac{9x^2 + 3ax + 3ax + a^2}{12ax}}\, dx = \sqrt{\frac{3x(3x+a) + a(3x+a)}{12ax}}\, dx$$

$$= \sqrt{\frac{(3x+a)^2}{12ax}}\, dx = \frac{3x+a}{\sqrt{12a}\sqrt{x}}\, dx$$

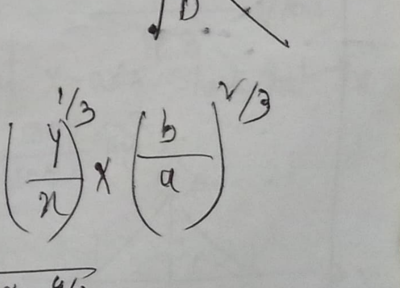
$$\therefore \frac{2}{\sqrt{12a}} \int_0^a \frac{3x+a}{\sqrt{x}}\, dx = \frac{2}{\sqrt{12a}} \int_0^a 3\sqrt{x} + \frac{a}{\sqrt{x}}\, dx$$

$$= \frac{2 \times 6}{3\sqrt{12a}} \left[x^{3/2}\right]_0^a + 2\times 2\left[\sqrt{x}\right]_0^a \times \frac{a}{\sqrt{12a}}$$

$$= \frac{4}{2\sqrt{3a}}\, a^{3/2} + \frac{2\sqrt{a}\sqrt{a}}{\sqrt{3}}$$

$$= \frac{2a}{\sqrt{3}} + \frac{2}{\sqrt{3}}a = \frac{4}{\sqrt{3}} \cdot a$$

6) The given curve



$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1 \quad \text{---(i)}$$

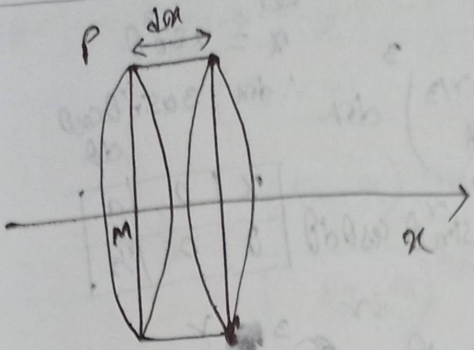
$$a^{-2/3} x^{2/3} + b^{-2/3} y^{2/3} = 1$$

$$\therefore \frac{2}{3} a^{-2/3} x^{-1/3} + \frac{2}{3} b^{-2/3} y^{-1/3} \frac{dy}{dx} = 0$$

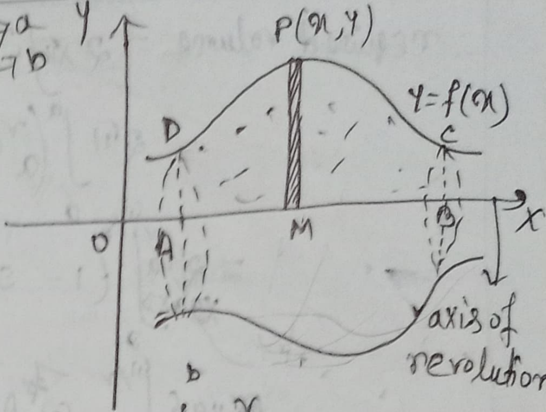
$$\text{or, } \frac{dy}{dx} = -\frac{a^{-2/3} x^{-1/3}}{y^{-1/3} b^{-2/3}} = \left(\frac{y}{x}\right)^{1/3} \times \left(\frac{b}{a}\right)^{2/3}$$

$$\therefore ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \frac{y^{2/3} b^{4/3}}{x^{2/3} a^{4/3}}}\, dx$$

Volume of revolution



$A \to a$
$B \to b$



$V = \pi (PM)^2 \, dx$
$= \pi y^2 \, dx$

$\therefore V = \int_{x=a}^{b} \pi y^2 \, dx$
$= \pi \int_a^b y^2 \, dx$

i) ~~axi~~ x axis as axis of revolution.

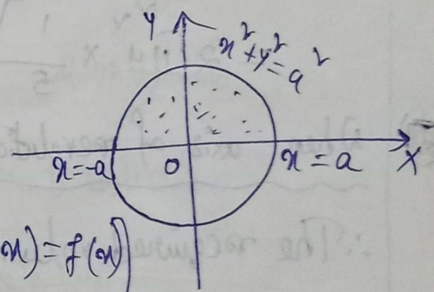
$$V = \pi \int_{x=a}^{b} y^2 \, dx$$

iii) y axis as axis of revolution,

$$V = \pi \int_{y=a}^{b} x^2 \, dy$$

1) Find the volume of a ~~sug square~~ sphere of radius a.

$\Rightarrow$ $\therefore$ Required volume, $V = \pi \int y^2 \, dx$



$= \pi \int_{-a}^{a} (a^2 - x^2) \, dx$

$= 2\pi \int_0^a (a^2 - x^2) \, dx \quad [\because f(-x) = f(x)]$

~~$= 2\pi \left[\frac{x}{2}(a^2 - x^2) + \frac{a^2}{2} \sin^{-1}\frac{x}{a} \right]_0^a$~~

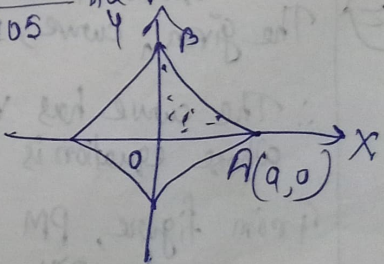
$= 2\pi a^2 [x]_0^a - \frac{2\pi}{3}[x^3]_0^a = 2\pi a^3 - \frac{2\pi}{3} a^3$

$= \frac{4\pi a^3}{3}$ (Ans)

(ii) An astroid $x^{2/3} + y^{2/3} = a^{2/3}$ revolves about x-axis. Show, that the volume of the solid thus generated each $\frac{32}{105}\pi a^3$.

$\Rightarrow$ The given curve $x^{2/3} + y^{2/3} = a^{2/3}$

or, $y = (a^{2/3} - x^{2/3})^{3/2}$ ——— (i)



$\therefore$ required volume $= 2 \times \pi \int y^2 \, dx$

Let, $x = a\sin^3\theta$

$\therefore dx = 3a\sin^2\theta\cos\theta \, d\theta$

$$= 2\pi \int_0^a \left(a^{2/3} - x^{2/3}\right)^3 dx$$

x	0	a
θ	0	$\pi/2$

$$= 2\pi a^2 \times a \times 3 \int_0^{\pi/2} (1 - \sin^2\theta)^3 \sin^2\theta \cos\theta \, d\theta$$

$$= 6\pi a^3 \int_0^{\pi/2} \cos^7\theta \sin^2\theta \, d\theta = 6\pi a^3 \cdot \frac{1\cdot 6\cdot 4\cdot 2}{9\cdot 7\cdot 5\cdot 3}$$

$$= \frac{32}{105}\pi a^3$$

~~$= 6\pi a^2 \int_0^{\pi/2} \cos^7\theta (1 - \cos^2\theta) d\theta$~~

$$= 6\pi a^2 \int_0^{\pi/2} \cos^3\theta \, d\theta - 6\pi a^2 \int_0^{\pi/2} \cos^5\theta \, d\theta$$

$$= 6\pi a^2 \left[\frac{2}{3\cdot 1} - \frac{4\cdot 2}{5\cdot 3\cdot 1}\right]^{\pi/2}$$

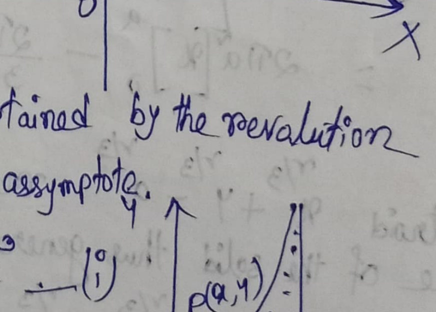
$$= 6\cdot 3\pi a^2 \times \frac{2}{3}\left[1 - \frac{4}{5}\right]$$

$$= 2\pi a^2 \times \frac{1}{5} = \frac{2}{5}\pi a^2$$

When axis of revolution other than x or y axis:—

$\therefore$ The required volume, V:

$$= \pi \int PM^2 \, d(AM)$$



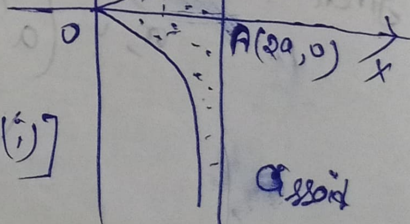
1) Find the volume of the solid obtained by the revalution of $y^2(2a-x) = x^3$ about it's assymptote.

$\Rightarrow$ The given equation of the curve, $y^2(2a-x) = x^3$ —— (i)

$\therefore$ The curve has vertical asymptote whose equation is $x = 2a$.

From figure, $PM = 2a - x$

$$AM = y = \frac{x^{3/2}}{(2a-x)^{1/2}} \quad [\text{by (i)}]$$



$$d(AM) = \frac{\left(\sqrt{2a-x}\cdot\frac{3}{2}x^{1/2} - x^{3/2}\cdot\frac{-1}{2\sqrt{2a-x}}\right)}{(2a-x)}\,dx$$

$$= \frac{1}{2}\,\frac{(2a-x)\,3x^{1/2} + x^{3/2}}{(2a-x)^{3/2}}\,dx$$

$$= \frac{1}{2}\,\frac{\sqrt{x}\{3(2a-x)+x\}}{(2a-x)^{3/2}}\,dx = \frac{x^{1/2}(3a-x)}{(2a-x)^{3/2}}\,dx$$

$\therefore$ required volume $= 2\pi\int (PM)^2\, d(AM) = 2\pi\int_{x=0}^{2a} \frac{(2a-x)^2(3a-x)\sqrt{x}}{(2a-x)^{3/2}}\,dx$

$$= 2\pi\int_0^{2a}\sqrt{x(2a-x)}\,(3a-x)\,dx$$

$x = 2a\sin^2\theta$

$dx = 4a\sin\theta\cos\theta\,d\theta$

x	0	$2a$
θ	0	$\pi/2$

$$= 2\pi\int_0^{\pi/2}\sqrt{2a\sin^2\theta\,(2a-2a\sin^2\theta)}\,(3a-2a\sin^2\theta)\,4a\sin\theta\cos\theta\,d\theta$$

$$= 2\pi\int_0^{\pi/2} 2a^3\sin^2\theta\cos^2\theta \times 4\,(3-2\sin^2\theta)\,d\theta$$

$$= 16\pi a^3\int_0^{\pi/2}(3-2\sin^2\theta)\sin^2\theta\cos^2\theta\,d\theta$$

~~$= 16\pi a^3\int_0^{\pi/2}(3-2\sin^2\theta)\sin^2\theta(1-\sin^2\theta)\,d\theta$~~

$$= 16\pi a^3\left[3\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta - 2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta\right]$$

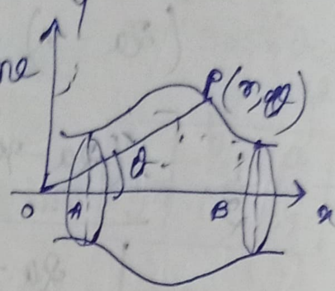
$$= 48\pi a^3\cdot\frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} - 32\pi a^3\,\frac{3\cdot 1\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}$$

$$= 3\pi^2 a^3 - \pi^2 a^3 = 2\pi^2 a^3 \text{ (Ans)}$$

Volume of revolution in polar co-ordinate system:-

i) When axis of revolution is initial line

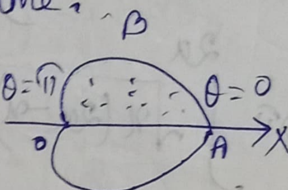
$$V = \frac{2\pi}{3}\int r^3 \sin\theta \, d\theta$$



ii) When axis of revolution is $\theta = \frac{\pi}{2}$

$$V = \frac{2\pi}{3}\int r^3 \cos\theta \, d\theta$$

1) Find the volume of revolution by revolving the Cardioid $r = a(1+\cos\theta)$ about the initial line.



⇒ given curve, $r = a(1+\cos\theta)$ —(i)

∴ required volume $= 2\times\frac{2\pi}{3}\int r^3 \sin\theta \, d\theta$

$$= \frac{2\pi}{3}\int_{\theta=0}^{\pi} a^3(1+\cos\theta)^3 \sin\theta \, d\theta$$

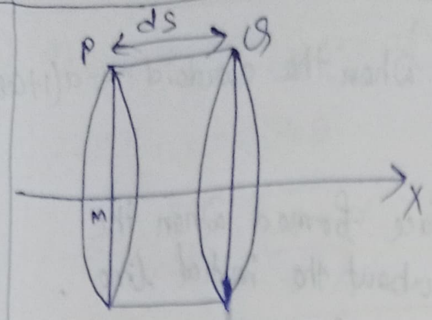
~~$= \frac{4\pi a^3}{3}\int (1+3\cos\theta+3\cos^2\theta+\cos^3\theta)\sin\theta\,d\theta$~~

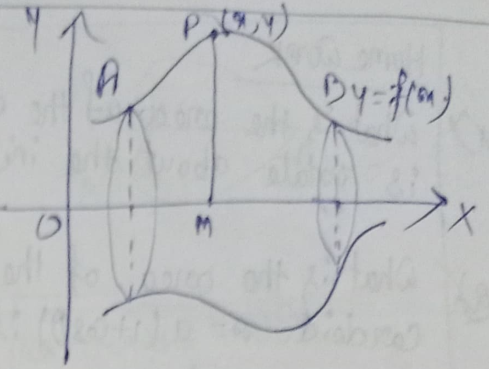
$$= \frac{2\pi a^3}{3}\int_{\pi}^{0} (1+\cos\theta)^3 \, d(1+\cos\theta)$$

$$= \frac{2\pi a^3}{3\times 4}\Big[(1+\cos\theta)^4\Big]_{\pi}^{0} = \frac{\pi a^3}{6}\left[(1+1)^4 - (1-1)^4\right]$$

$$= \frac{\pi a^3}{6}\times 16 = \frac{8\pi a^3}{3}$$

Surface of revolution





i) x-axis as axis of revolution :— $S = 2\pi \int y\, ds$

ii) ~~Where~~ a) for Cartesian curve, $y = f(x)$, $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

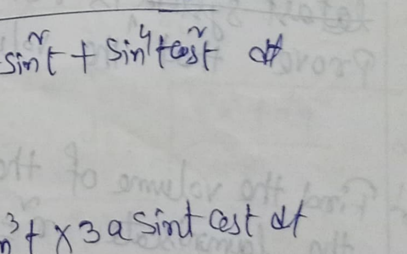
b) for Polar curve, $r = f(\theta)$, $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

c) for Parametric curve, $x = x(t)$, $y = y(t)$, $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

iii) y axis as axis of revolution :— $S = 2\pi \int x\, ds$

where $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

1) The arc of the astroid $x = a\cos^3 t$, $y = a\sin^3 t$ revolves about x-axis. Find the surface of the solid ~~thus~~ generated thus.



⇒ given that,

$x = a\cos^3 t$, $y = a\sin^3 t$

$\therefore \frac{dx}{dt} = -3a\cos^2 t \sin t$, $\frac{dy}{dt} = 3a\sin^2 t \cos t$

$\therefore ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = 3a\sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t}\, dt$

$= 3a \sin t \cos t\, dt$

$\therefore$ required surface area $= 2 \times 2\pi \int_{\pi/2}^{0} a\sin^3 t \times 3a \sin t \cos t\, dt$

$= 12\pi a^2 \int_{\pi/2}^{0} \sin^4 t \cos t\, dt$

When, $x = 0$
$\cos^3 t = 0$
$\Rightarrow \cos t = 0$
$\Rightarrow t = \frac{\pi}{2}$

When $x = a$
$\cos t = 1$
$t = 0$

$= 12\pi a^2 \int_{\pi/2}^{0} (1 - \cos^2 t)^2 \cos t\, dt = 12\pi a^2 \int_{\pi/2}^{0} (1 + \cos^4 t - 2\cos^2 t) \cos t\, dt$

$= 12\pi a^2 \int_{\pi/2}^{0} \cos t + \cos^5 t\, dt - 2\cos^3 t\, dt = 12\pi a^2 \int_{0}^{\pi/2} (2\cos^3 t - \cos^5 t - \cos t)\, dt$

$= 12\pi a^2 \left[2 \cdot \frac{3}{4} \cdot \frac{1}{2} \times \frac{\pi}{2} - \frac{4}{5} \times \frac{2}{3} \times 1 - 1\right] = 12\pi a^2 \left[\frac{3\pi}{8} - \frac{23}{15}\right]$

$\frac{8}{15}$ $\frac{21}{15}$

Home work :

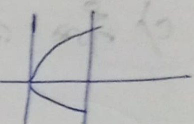
1) What is the area of the entire surface when the cardoid $r = a(1+\cos\theta)$ is rotate about the initial line.

2) What is the area of the entire surface formed when the cardoid $r = a(1+\cos\theta)$ is revolved about the initial line.

3) Find the surface area of the elipsoid formed by the revolution of the elips $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ round the minor axis.

4) Find the volume of the solid of revolution formed by the rotation of the Parabola $y^2 = 4ax$ bounded by the section $x = a$ about x axis.

$\pi \int_0^a y^2 dx \rightarrow 2\pi a^2$



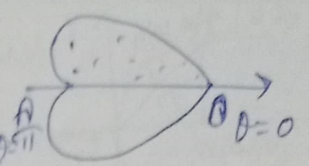
5) Find the volume of the solid generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ about x axis. $\rightarrow 5\pi^2 a^3$.

6) Find the volume of the solid generated by the revolution of the curve $y = \frac{a^3}{a^2+x^2}$ about its asymptotes

7) Find the volume of the solid generated by the revolution of the curve $y^2 = 4ax$ from vertex to one extremity of the ~~latast system~~ latus rectum is revolved about the corresponding chord. Prove that the volume of the spindle form is $\frac{2\sqrt{5}}{75}\pi a^3$

8) Find the volume of the solid generated by the revolution of the lamniscate $r^2 = a^2\cos 2\theta$ about $\theta = \frac{\pi}{2}$ $\rightarrow \frac{1}{4\sqrt{2}}\pi^2 a^3$.

1) The given curve, $r = a(1+\cos\theta)$



$\therefore \frac{dr}{d\theta} = -a\sin\theta$

$\therefore ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

$= \sqrt{a^2 + a^2\cos^2\theta + 2a^2\cos\theta + a^2\sin^2\theta}\, d\theta = \sqrt{2a^2 + 2a^2\cos\theta}\, d\theta$

$= a\sqrt{2\cdot 2\cos^2\theta/2}\, d\theta = 2a\cos\frac{\theta}{2}\, d\theta$

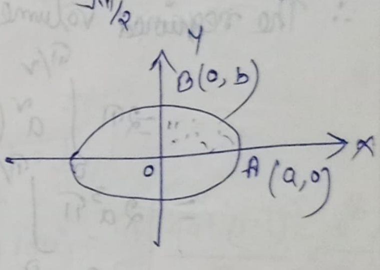
$\therefore$ The required surface of area, $S = 2\pi\int_0^{\pi} y\, ds$

$= 2\pi\int_0^{\pi} r\sin\theta\, ds = 2\pi a\int_0^{\pi}(1+\cos\theta)\sin\theta\, 2a\cos\frac{\theta}{2}\, d\theta$

$= 4\pi a^2\int_0^{\pi} 2\cos^3\theta/2\; 2\sin\theta/2\cos\theta/2\, d\theta$

$= 2\times 16\pi a^2\int_0^{\pi/2}\sin\theta/2\cos^4\theta/2\, d\theta = \frac{32\pi a^2}{5}\left[\cos^5\theta/2\right]_{\pi/2}^{0} = \frac{32\pi a^2}{5}.$

3) The given curve,



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ —(i) $\quad b^2x^2 + a^2y^2 = a^2b^2$, $\quad x^2 = \frac{a^2b^2 - a^2y^2}{b^2}$

$\therefore \frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

or, $\frac{dy}{dx} = -\frac{xb^2}{a^2y}$

~~$\therefore ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1+\frac{x^2b^4}{a^4y^2}}\, dx$~~ $= \sqrt{\frac{x^2b^4 + y^2a^4}{a^4y^2}}\, dx$

$\therefore ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1+\frac{a^4y^2}{x^2b^4}}\, dy = \sqrt{\frac{x^2b^4 + a^4y^2}{x^2b^4}}\, dy$

$= \sqrt{\frac{\frac{a^2b^2 - a^2y^2}{b^2}\times b^4 + a^4y^2}{\frac{a^2b^2 - a^2y^2}{b^2}\times b^4}}\, dy = \sqrt{\frac{a^2b^4 - a^2b^2y^2 + a^4y^2}{a^2b^4 - a^2b^2y^2}}\, dy$

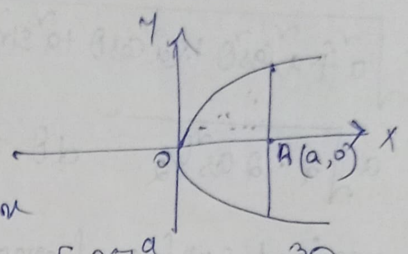
$\therefore$ The required area, $S = 2\int_0^b 2\pi x\, ds = \frac{4\pi}{b}\int_0^b \sqrt{\frac{(a^2b^2 - a^2y^2)(a^2b^4 + a^4y^2 - a^2b^2y^2)}{a^2b^4 - a^2b^2y^2}}\, dy$

$= \frac{4\pi}{b}\int_0^b \sqrt{\frac{(b^2-y^2)(b^4 - b^2y^2 + a^2y^2)}{b^4 - b^2y^2}}\, dy = \frac{4\pi}{b}\int_0^b \sqrt{\frac{(b^2-y^2)(b^2 - y^2b^2 + a^2y^2)}{b^2(b^2-y^2)}}\, dy$

$= \frac{4\pi}{b^2}\int_0^b \sqrt{b^2 - y^2b^2 + a^2y^2}\, dy$

4) The given curve,

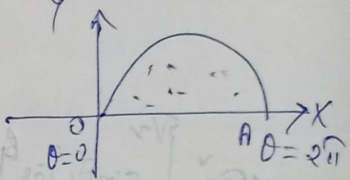
$y^2 = 4ax$



$\therefore$ The required volume $= \pi \int_0^a y^2 \, dx$

$= \pi \int_0^a 4ax \, dx = \frac{4a\pi}{2} \left[x^2 \right]_0^a = 2a^3\pi$

5) The given curve,



$x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$

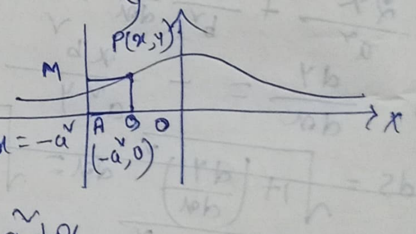
$\therefore$ The required volume $= \pi \int_0^{2\pi} y^2 \, dx$

$= 2\pi \int_0^{\pi/2} a^2 (1-\cos\theta)^2 \, a(1-\cos\theta) \, d\theta$

$= 2a^3\pi \int_0^{\pi/2} (1-\cos\theta)^3 \, d\theta = 2a^3\pi \int_0^{\pi/2} 8 \sin^6 \theta/2 \, d\theta$

$= 16a^3\pi \int_0^{\pi/2} \sin^6 \theta/2 \, d\theta = 5\pi^2 a^3$

6) The given curve, $y = \frac{a^3}{a^2 + x}$



or, $y(a^2 + x) = a^3$

The coefficient of highest power of y is $a^2 + x$.

$\therefore$ The vertical asymptote is $x = -a^2$

From the figure, $AM = y$ and $PM = AO = OA - OO$
$= -a^2 - x$

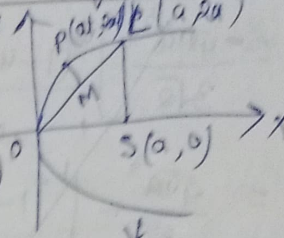
$\therefore d(AM) = d\left(\frac{a^3}{a^2+x}\right) = -\frac{a^3}{(a^2+x)^2} dx$

$\therefore$ The required volume $= \pi \int_0^{-a^2} (PM)^2 \, d(AM) = \pi \int_0^{-a^2} (a^2+x)^2 \frac{a^3}{(a^2+x)^2} dx$

$= -\pi \int_0^{-a^2} a^3 \, dx = -\pi a^3 \left[x \right]_0^{-a^2}$

$= -\pi a^3 \left[-a^2 \right] = \pi a^5$

7) The given curve,
$y^2 = 4ax$



Let, $(at^2, 2at)$
P is a point on the parabola, where.

The equation of OL, $y = 2x$

Now, The length of $PM = \frac{2at - 2at^2}{\sqrt{5}} = \frac{2at}{\sqrt{5}}(1-t)$

$\therefore (OM)^2 = (OP)^2 - (PM)^2 = (at^2)^2 + (2at)^2 - \frac{4a^2t^2}{5}(1-t)^2$

$= a^2t^4 + 4a^2t^2 - \frac{4a^2t^2(1+t^2-2t)}{5} = \frac{5a^2t^4 + 20a^2t^2 - 4a^2t^2 - 4a^2t^4 + 8a^2t^3}{5}$

$= \frac{a^2t^4 + 16a^2t^2 + 8a^2t^3}{5} = \frac{a^2t^2}{5}(t^2 + 16 + 8t) = \frac{a^2t^2}{5}(t+4)^2$

$\therefore (OM) = \frac{at}{\sqrt{5}}(t+4)$, $d(OM) = \frac{a}{\sqrt{5}}(2t+4)dt$

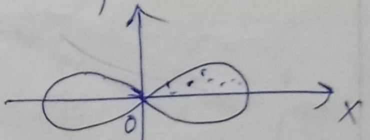
$= \frac{a}{\sqrt{5}}(t^2 + 4t)$

~~When, t=~~ at O $t=0$ and at L $t=1$

$\therefore$ The required volume $= \pi \int_0^1 (PM)^2 \, d(OM) = \pi \int_0^1 \frac{4a^2t^2(1-t)^2}{5} \times \frac{a}{\sqrt{5}}(2t+4) \, dt$

$= \frac{8\pi a^3}{25\sqrt{5}} \int_0^1 t^2(1-t)^2(t+2)\,dt = \frac{2\sqrt{5}}{75}\pi a^3$. (Proved)

8) The given curve, $r^2 = a^2 \cos 2\theta$



$\therefore$ The required volume $= 2 \times \frac{2\pi}{3} \int_0^{\pi/4} r^3 \cos\theta \, d\theta$

$= \frac{4\pi a^3}{3} \int_0^{\pi/4} \cos 2\theta \sqrt{\cos 2\theta} \, \cos\theta \, d\theta$

$= -\frac{4\pi a^3}{3} \int_1^0 x^2 \cdot x \sqrt{\frac{x^2+1}{2}} \frac{x\,dx}{\sqrt{1-x^4}}$

$= \frac{4\pi a^3}{3\sqrt{2}} \int_0^1 x \sqrt{\frac{1+x^2}{(1+x^2)(1-x^2)}} \, dx = \frac{4\pi a^3}{3\sqrt{2}} \int_0^1 \frac{x^3 \, dx}{\sqrt{1-x^2}}$

$= \frac{4\pi a^3}{3\sqrt{2}} \int_0^1 \frac{x^2 \cdot x \, dx}{\sqrt{1-x^2}}$

Let,

$\cos 2\theta = x^2$

$-2\sin 2\theta \, d\theta = 2x\,dx$

$d\theta = -\frac{x\,dx}{\sqrt{1-x^4}}$

$\cos^2\theta - \sin^2\theta = x^2$

or, $2\cos^2\theta - 1 = x^2$

or, $\cos\theta = \sqrt{\frac{x^2+1}{2}}$

θ	0	$\pi/4$
x	1	0

$$= -\frac{4\pi a^3}{3\sqrt{2}} \int_1^0 \frac{(1-u^2)\,u\,du}{u}$$

$$= \frac{4\pi a^3}{3\sqrt{2}} \int_0^1 (1-u^2)\,du$$

$$= \frac{4\pi a^3}{3\sqrt{2}} \left[u - \frac{u^3}{3}\right]_0^1$$

$$= \frac{4\pi a^3}{3\sqrt{2}} \times \frac{2}{3} = \frac{8\pi a^3}{9\sqrt{2}}$$

Let, $1-x^2 = u^2$

$-2x\,dx = 2u\,du$

or, $x\,dx = -u\,du$

$\therefore x^2 = 1-u^2$

x	0	1
u	1	0

$$= \frac{4\pi a^3}{3\sqrt{2}} \int_0^1 \sqrt{\frac{x^2}{1-x^2}}\,dx = \frac{4\pi a^3}{3\sqrt{2}} \int_0^1 \frac{(1-x^2-1)}{\sqrt{1-x^2}}\,dx$$

$$= \frac{4\pi a^3}{3\sqrt{2}} \int_0^1 \sqrt{}$$

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

x	0	1
θ	0	$\pi/2$

$$= \frac{4\pi a^3}{3\sqrt{2}} \int_0^{\pi/2} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\,\cos\theta\,d\theta$$

$$= \frac{4\pi a^3}{3\sqrt{2}} \int_0^{\pi/2} \sin\theta\,d\theta = -\frac{4\pi a^3}{3\sqrt{2}}\left[\cos\theta\right]_0^{\pi/2} = \frac{4\pi a^3}{3\sqrt{2}}$$